Ideas for Nifty Problem Development
SIGCSE 2005 Nifty Discrete Math Workshop

- Develop simplified versions of the problem for students to start with
- Create a sequence of smaller/incremental problem solving activities
- Present generalized versions of the problems where appropriate
- Classify/categorize the problem
- Develop declarative (functional or logic based) solutions where appropriate
- Identify Keywords Type of problem. Solution. Pedagogical Notes. Historical Notes, Learning Outcomes and Prerequisite Knowledge (See example “Coin Problem” attached )

1. There's an isolated island with \( k \) natives living on it. The natives cannot communicate with each other in any way. All the natives have blue eyes. One night, the devil comes, and turns \( n > 0 \) of these natives from blue eyes to having brown eyes. All natives know that the devil has turned at least one of them from blue eyes to brown eyes. Each night the natives gather in a circle and look at each other. However, if a native has deduced that they have brown eyes, they don't come to any future gatherings.

Do any of the natives not come to future gatherings? If so, how long does it take until all \( n \) of the brown-eyed don't come to future gatherings? (NOTE: There are no mirrors or any reflecting surfaces of any kind on this island, so the natives are unable to see the color of their own eyes.) HINT: Try putting yourself in the place of a native. What information can you deduce each night? Think Inductively!!

2. In a prison all \( 1000 \) jail cells are in a row. A jailer, carrying out the terms of a partial amnesty, unlocked every jail cell in the row. Next, starting from the first he locked every second cell. Then, starting from the first, he turned the key in every third cell, locking those which were open and opening those which were locked. He continued this pattern of locking and unlocking every \( n^{th} \) cell on the \( n^{th} \) trip until eventually he was done. Those prisoners whose cells eventually remained open were allowed to go free. In which cells were the lucky prisoners? How many times did he have to walk down the row of cells performing this procedure?

3. Consider the problem "Given a set of \( N \) points in the plane, find the smallest circle containing all \( N \) points." Joe, a computer scientist major, claims this is easily solved by finding the two points which are furthest apart, and then drawing a circle centered midway between these two points whose diameter is the distance between the two points. Is Joe right or wrong?
4. You and ninety-nine space travelers have survived a crashed on a small moon in the Quadulus system. There you found three spaceships, the MG, the NS, and the LB. Information on the moon warns you that you need 900 tons of fuel to get to safety and that the three spaceships have 500, 750, and 1000 of fuel and capacities of 50, 100, and 150 passengers, but they don't know which ship has how much fuel or how many each holds. But you do know that

- (a) MG carries more passengers than the ship with 750 tons of fuel.
- (b) LB has more fuel than the 50 passenger ship.
- (c) The 150 passenger ship has more fuel than the LB.

Can you and your friends get to safety? Which ship should you take and what is its fuel and passenger capacity? Give a brief statement to support your answer.

Answer: MG, 150 people, 1000 tons

5. Bill, Sue and Jack are suspected of civil disobedience. They testify under oath as follows:

Bill says: “Sue is guilty and Jack is innocent.”

Sue says: “If Bill is guilty, then so is Jack.”

Jack says: “I am innocent, but at least one of the others is guilty.”

Assuming that a person is either innocent or guilty, but not both, answer the following questions.

- (a) Can everyone's testimony be true?
- (b) The testimony of one of the suspects follows from that of another. Which from which?
- (c) Assuming that everyone is innocent, who committed perjury (who lied)?
- (d) Assuming that everyone's testimony is true, who is innocent and who is guilty?
- (e) Assuming that the innocent persons told the truth and the guilty ones told lies, who is innocent and who is guilty.
6. You have a circle of $2^n$ lights, for $n \geq 0$. Each light has two possible states: on or off. The lights synchronously change (i.e., change simultaneously or at the same time) their state according to the following rule: If a light is off, then the next light (in clockwise order) switches its state. If a light is on, then the next light (in clockwise order) does not switch its state.

   a. For the circle of 8 lights below, give the states for the first two synchronous changes. Here 0 is on and X is off.

   ![Diagram of 8 lights]

   b. Given an arbitrary initial state of a circle of $2^n$ lights after some finite amount of time will all lights be on, off or no stable state is achieved?

   c. Prove your answer to (b) is correct. That is, after a finite number of synchronous changes all lights will be on, off or no stable state is achieved.

7. A digital number display, such on a digital clock, uses 7 light bars, as shown below to represent the 10 digits 0, 1, 2, ..., 9.

   ![Diagram of 7 light bars]

   a. If any one of these light bars burns out, can the 10 digits still be distinguished?

   b. What is the minimal number of light bars which can burn out for the digits to be indistinguishable? Which bars are these?

8. There are 20 people at a party and 48 different pairs of these people are acquainted? (Here acquainted is defined as Person 1 knows Person 2 and Person 2 knows Person 1) Show that there exists some person at the party with 4 or fewer acquaintances?
9. Simon Singh was at Butler University Jan 2005 and presented a problem upon which the following is loosely based. Anything nifty here? Any interesting simplifications or generalizations?

Consider tossing a coin three times in a row.

a. How many different sequences of 3 heads and tails are possible?

b. List all these possible sequences using T for tails and H for heads.

c. Here is a two person game. The first person guesses a possible 3 toss outcome sequence. Then the second person selects a 3 toss outcome. The coin is tossed over and over until someone’s outcome results. That person wins.

d. If the first person selected H H H what sequence should the second person select to maximize their chance of winning? (answer: T H H why?)

e. For (d) above what it the first and second persons probability of winning?

f. If you were the first person playing this game and understood the mathematical aspects of this game, what sequence would you pick to maximize your chances of winning?

10. A jeep has a gas tank which holds a certain quantity of gasoline. Initially the jeep is parked at the edge of the Sahara desert with an empty gas tank. At the oasis there is a fuel dump containing n cans of gasoline, where each can holds exactly the same amount as the gas tank of the jeep. Assume that the driver may create new fuel dumps by transporting cans into the desert but that the jeep can carry only one can at a time in addition to the fuel in its tank. Also assume that the driver is only allowed to fill the tank when it is empty. What is the farthest point into the desert that the jeep can reach assuming all n cans of gasoline are used? Hint: First try some examples for specific values of n. Learn (inductively) from them.

Helpful notation: The distance the jeep can go on one tankful is d. Define a function f as follows: Assuming that there are n cans at the start, let f(n) be the multiplier of d corresponding to the maximum distance the jeep can travel if there are n cans to start with. For instance, if there are 2 cans, the jeep would fill up with one, load the other, travel d units, fill up with the other, and travel an additional d units. Thus, the jeep would travel 2d units, and f(n) would be 2.

A good but non-quite-optimal solution is the following: Fill the tank with one can, and transport all the remaining cans to a location as far as possible from the initial point. Then repeat the process (think recursively!) until all the gas has been used up. The answer would be: f(n) = f(n-1) + 1/(2n-3), with f(1) = 1 or, in closed form, f(n) = 1 + sum from i=2 to n of (1/(2i-3))
11. In a round-robin tournament each team plays every other team exactly once. The win/loss results of a round-robin tournament can be represented using a directed graph (eg. 1 → 2 might represent Team 1 beats Team 2). Some potential questions are: With n > 1 teams, how many games/matches are there? Can there be a cycle in this graph? If so, what are the characteristics of this cycle? Can a clear tournament winning team always be determined? If so, how? Same for a losing team? Can a ranking of teams be determined? What properties of this ranking are there? See Epp, 3rd edition, Page 234)

12. Consider a 3×3 grid consisting of 6 on/off switches and 9 lights as shown below. A light is lit only if its corresponding column and row switch are on. Find invariant conditions for which it is possible and/or impossible to determine the state of the switches from the state of the lights.

Col Sw 1  Col Sw 2  Col Sw 3
|     |     |     |
|     |     |     |
|     |     |     |
o-----o--------o------  Row Switch 1
|     |     |     |
|     |     |     |
o-----o--------o------  Row Switch 2
|     |     |     |
|     |     |     |
o-----o--------o------  Row Switch 3

Here is a sample from the 2004 MAA PREP Workshop regarding one potential format you can use for nifty problems.
The Coin Problem
Elana Epstein
St. Joseph's College

Aaron has 36 cents, Betty has 52 cents, and Carl has 89 cents. All of their money is in the highest amount possible. For example, 47 cents would consist of one quarter, 2 dimes, and 2 pennies.

a) If Aaron and Betty combined their money then what coins do they have together (and how many of each)?
b) What coins do Betty and Carl have in common?
c) What coins does Carl have that Aaron doesn't?

Keywords: Sets, union, intersection, set difference

Type: Homework problem

Solution: a) 3 quarters, 1 dime, and 3 pennies  
b) 2 quarters and 2 pennies  
c) 2 quarters and 3 pennies

Pedagogical Notes: This is a very simple problem, but it could provide a good example for when students are first learning sets. Most set problems are abstract, whereas this problem deals with something that students use everyday. They shouldn't have any problems doing this example, and then you can refer back to it when doing the union, intersection, and difference definitions.

Learning Outcomes: Prepares students to learn about sets.

Prerequisite Knowledge: none.