The Two Color Theorem

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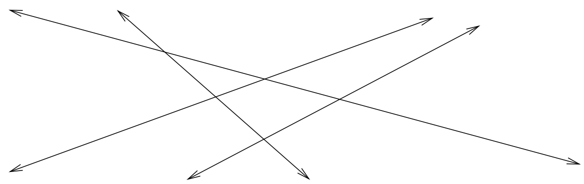
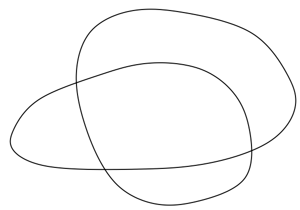
The Four Color Theorem is a great example of a result that is simple to state, but very hard to prove. Yet under certain conditions, it is possible to show that two colors suffice:

I. Consider any finite collection of (infinite) straight lines in the plane.



These lines divide the plane into regions, forming a map. Prove that this map can be two-colored.

II. A *closed curve* in the plane is a continuous loop, possibly with self-intersections (i.e., an immersion of S^1). For example, a closed curve can be drawn by placing a pencil at point X, drawing any curve without lifting the pencil, and finishing the curve at point X.



A closed curve divides the plane into regions, forming a map. Prove that this map can be two-colored.

Keywords: induction, structural induction, graphs, colorability, map coloring, Four Color Theorem, trees, decision trees, parity check

Type: lecture example, homework problem, in-class group assignment, research project

Solutions: There are many ways to prove I and II. Descriptions of two solutions for each problem follow. Detailed solutions can be found below.

I. Straight lines in the plane:

1. Use induction on the number of lines.
2. Do a parity check on each region.

II. Closed curve:

1. Use induction on the number of crossings.
2. Represent the map as a planar graph. Use induction on the number of faces.

Pedagogical Notes: Instructors who do not wish to spend a lot of time on this problem should probably stick to version I. However, version II is perhaps a more surprising result, and presentation/discovery of either solution should be rewarding for mathematically oriented students.

Solution I.1 is the easiest and most intuitive, and gives a good example of an inductive argument in a simple geometric setting. Solution I.2 avoids induction, and is quite elegant, especially given the difficulty of problem II.

Solution II.1 involves some intuition about the topology of curves in the plane. To develop this intuition, students might want to experiment with loops of string (possibly even knotted). The trickiest part of the proof involves showing that a smoothing can always be chosen so that the curve remains connected. This property (and the basis of the inductive argument) can be discovered by constructing a binary decision tree, where each branch gives the result of smoothing a vertex in two different ways. This decision tree is related to the calculation of the Kauffman bracket polynomial, an important invariant in knot theory. Interested students could investigate related topics in knot theory in [K] or [A].

The only topological insight needed for solution II.2 is that each intersection yields a four-sided face, and that all faces of the graph are obtained in this way. Granting this, the graph theory argument is standard, though perhaps challenging for most students. This approach gives the most general characterization of a two-colorable map, and invites investigation of the converse, leading to connections with graph walking.

Learning Outcomes: This problem provides some interesting examples of inductive arguments (and/or structural induction), and can be used to illustrate inductive thinking in a geometric/topological setting. It also shows that some continuous processes have underlying discrete properties. Solution I.2 shows that sometimes it can be simpler to use a non-inductive argument, even in a situation where it seems natural to use induction. Solution II.2 is a good application for some standard results in graph theory.

Connections: graph theory, structural induction, decision trees, knot theory, topology, Kauffman bracket polynomial, graph walking, Four Color Theorem, parity check

Prerequisite Knowledge: Definition of graph (for solution II.2).

References:

[A] Colin Adams, *The Knot Book*, Freeman, 1994.

[G] Martin Gardner, *Sixth book of mathematical games from Scientific American*, Freeman, 1971.

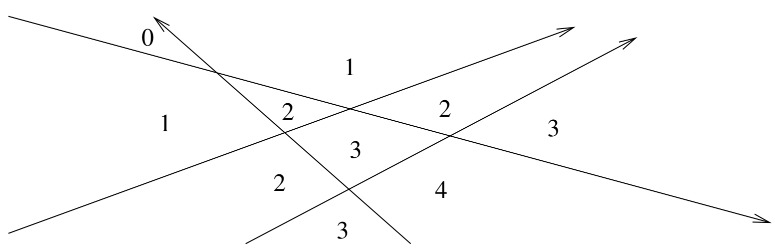
[K] Louis Kauffman, *On Knots*, Princeton University Press, 1987.

Notes on Suggested Solutions:

I. Straight lines in the plane:

1. Use induction on the number of lines. If there are no lines, color the entire region white. Now let $n > 0$. Suppose as inductive hypothesis that any collection of $n-1$ lines can be two-colored. Given a collection of n lines, remove any line l and two-color the resulting map, by inductive hypothesis. Now put l back. Reverse the coloring on one side of the l. Each side of l will be correctly two-colored, and it remains to show that any two regions whose border lies on l have opposite colors. But any two such regions must have been the same region when l was removed, so the reversal of colors guarantees that the new regions will have opposite colors. □

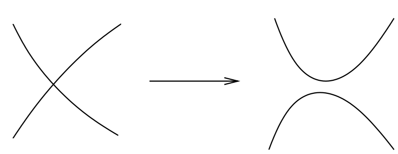
2. (This solution is adapted from [G], which considers a map formed by a collection circles in the plane.) Choose an orientation (i.e., a preferred direction) on each line. For each region R, count the number $k(R)$ of lines l such that R is on the right side of l.



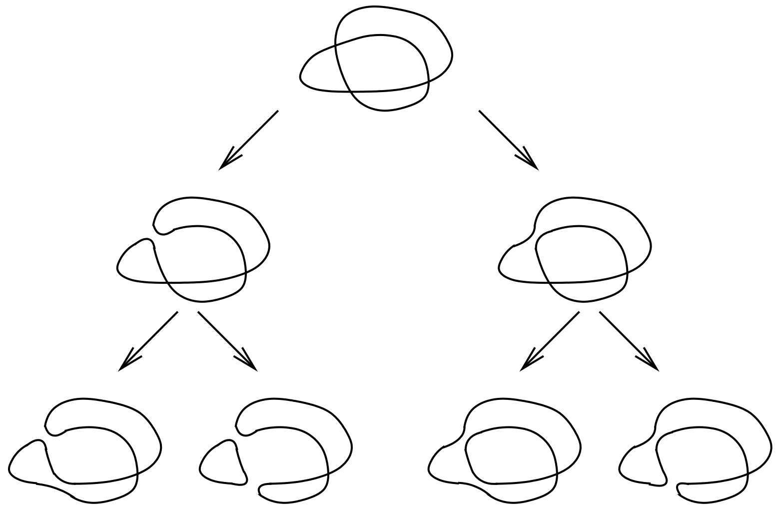
If $k(R)$ is even, color R black; otherwise, color R white. By construction, this is a two-coloring, since regions on opposite sides of any boundary will have different colors. □

II. Closed curve:

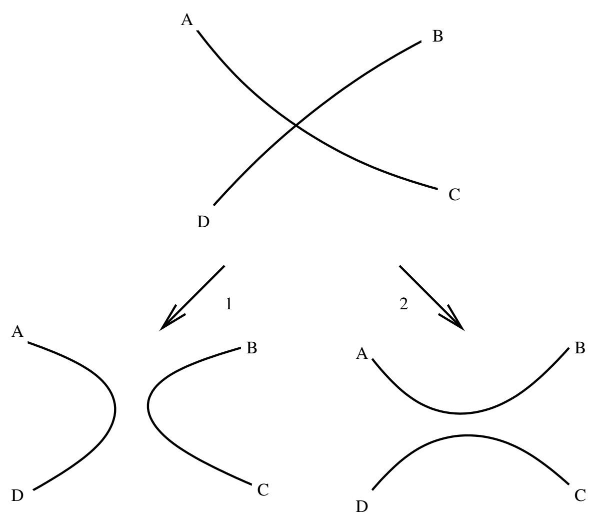
1. (Uses topological intuition.) For simplicity, assume that the curve is in general position (i.e., no multiple crossings, no self-tangents). A crossing can be removed by performing a *smoothing*:



At any given crossing, there are two ways to perform a smoothing. Therefore, you can represent the process of smoothing all the crossings as a binary decision tree. For example, for the curve given in the problem introduction, the first two levels of one choice of tree would be as follows.

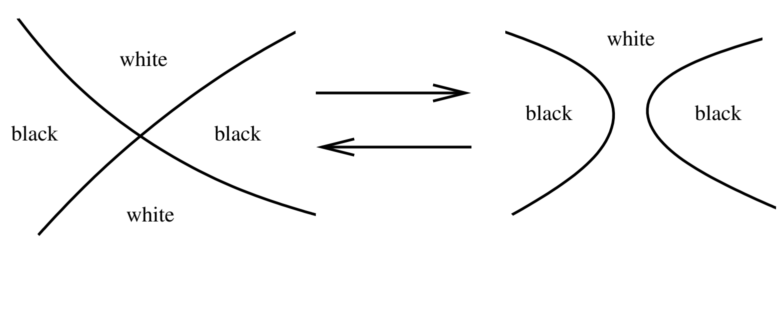


Notice that a smoothing may change the number of components of the curve, but it is always possible to choose a smoothing that leaves the curve connected. To see this, consider any crossing in any given curve:



Suppose, without loss of generality, that when traversing the original curve starting at point A, you encounter points A, C, B, D, and then A, in order. This must happen (up to a choice of labels) because the original curve is closed and connected. Then, since C is connected to B directly (without passing through A), smoothing #2 must not disconnect the curve.

Now we can prove that any closed curve map can be two-colored by induction on the number of crossings. If there are no crossings, simply color the inside of the curve black and the outside white. (Note that we are using the Jordan Curve Theorem here.) Let $n > 0$. Suppose, as inductive hypothesis, that any closed curve graph with $n-1$ crossings can be two-colored. Given a curve with n crossings, perform a smoothing on any crossing (and choose the smoothing so that the curve remains connected). By inductive hypothesis, the smoothed curve can be two-colored. The following picture shows how to color the n-crossing curve using the coloring of the $(n-1)$-crossing curve. □



2. (Uses graph theory.) Given any closed curve map, construct a planar graph G as follows: place one vertex in each region, and connect any two vertices whose regions share a common border (i.e., take the dual graph of the graph whose vertices are the crossings of the curve, and whose edges follow the curve).
Claim: All the faces of G have an even number of edges.
Proof of claim: By duality, each face of G contains one crossing, and the number of edges of each face is the number of paths to the crossing. Since the curve is closed, each crossing has an even number of paths to it, because the curve cannot terminate at a crossing. (Note that if we assume general position, then each face of G must have four edges.) $\square$
It now suffices to prove the following.
Lemma: If every face of a planar graph has an even number of edges, then the graph is two-colorable.
Proof: We proceed by induction on the number of faces in the graph. If the graph has no faces, then it has no cycles, so it is bipartite, i.e., two-colorable. Let $n > 0$. Suppose as inductive hypothesis that every planar graph with $n - 1$ faces, all of which have an even number of edges, is two-colorable. Let G be a planar graph, and let G have n faces, all of which are even-edged. Choose any face $X_1 X_2 \cdots X_{2k}$, and form a new graph G' by removing edge $X_1 X_2$ from G. By construction, G' has $n - 1$ faces, so it can be two-colored. Since the trail $X_2 X_3 \cdots X_{2k} X_1$ can be two-colored, the colors must alternate, so X_2 and X_1 must have different colors. Hence the two-coloring of G' is a valid two-coloring of G. $\square$