Looping and The Figurate Numbers

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A Nifty Application of Discrete Mathematics
to Computer Science

An ancient method of counting involved representing numbers with pebbles, which could be arranged into geometric figures, thereby associating certain numbers with specific geometric shapes. For example, there are linear numbers, triangular numbers and pyramidal numbers, which give the number of pebbles or dots in certain sequences of lines, triangles and pyramids respectively. Initially these figurate numbers were used for discrete approximations to areas or volumes of the figures they represent, but then found applications in several branches of mathematics, including finite summations, the computation of probabilities, and the counting of arrangements of objects other than pebbles. This project offers an application of these numbers to counting iterations of a certain type of loop that occurs in computer programming.

Let's first investigate some simple properties of the figurate numbers known to the ancients. Although in antiquity, mathematics was written verbally and pictorially, we will use modern algebraic equations along with the figures which visually capture the meaning of these numbers.

(a) The linear numbers are given by the number of dots in the following sequence of line segments:

First Line Second Line Third Line Fourth Line

Let $L_1 = 1$, $L_2 = 2$, $L_3 = 3$, etc. be the sequence of linear numbers. Although the value of L_j is apparent, state how L_{j+1} is computed from L_j, $j \geq 1$.

(b) The triangular numbers, allegedly studied by Pythagoras of Samos (c. 572–497 B.C.E.) and his school of Pythagoreans [1, p. 49], are given by the number of dots in the following sequence of triangles:

First Triangle Second Triangle Third Triangle Fourth Triangle

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Let $T_1 = 1$ be the first triangular number, $T_2 = 3$ the second, $T_3 = 6$ the third, etc. Compute T_4 and T_5. Explain how T_{j+1} is computed from T_j, $j \geq 1$, both geometrically and algebraically. Express this relation with an appropriate linear number which forms the base of T_{j+1}.

By placing two triangles representing T_4 side-by-side to yield a rectangle, and then counting the dots in this rectangle, explain why $2(T_4) = (4)(5)$. Use a similar geometric argument to find a formula for $2(T_j)$ in terms of j. Solving for T_j yields the modern formula for the jth triangular number.

(c) The pyramidal numbers, P_j, $j \geq 1$, are given by counting the dots in certain pyramids, constructed according to the following recursive procedure. The first pyramid consists of one dot, so that $P_1 = 1$. The second pyramid has the second triangle as its base, and one dot (the first pyramid) as its apex. Sketch this pyramid, and explain why $P_2 = 4$. The third pyramid is formed by placing the second pyramid atop the third triangle. Sketch the third pyramid, and compute P_3, the number of dots it contains. In general, the $(j+1)$st pyramid is formed by placing the jth pyramid atop the $(j+1)$st triangle. Use this geometric description to find an equation relating P_{j+1}, P_j, and T_{j+1}, $j \geq 1$. Find the values of P_4 and P_5.

(d) To the ancient Greeks, nothing existed beyond the third dimension, so their study of figurate numbers did not go beyond solid geometry. By the seventeenth century, however, Pierre de Fermat (1601–1665) writes of a triangulo-triangle [2, p. 36], which is a four-dimensional analogue of a pyramid, known today as a four-dimensional tetrahedron or simplex. The first triangulo-triangle is one dot, while the second has the second pyramid as its base, and one point as its apex in the fourth dimension. The third triangulo-triangle consists of the second triangulo-triangle atop the third pyramid, while the $(j+1)$st triangulo-triangle is constructed by placing the jth triangulo-triangle atop the $(j+1)$st pyramid. Let R_j, $j \geq 1$, denote the number of dots in the jth triangulo-triangle. Compute R_1, R_2 and R_3. Find an equation relating R_{j+1}, R_j and P_{j+1}.

(e) The construction of the figurate numbers for higher dimensions should now be clear. Let $F(j, m)$ denote the jth figurate number of the m-dimensional simplex (triangle, pyramid, etc.). In our previous notation

$$L_j = F(j, 1), \quad T_j = F(j, 2), \quad P_j = F(j, 3), \quad R_j = F(j, 4),$$

for integers j with $j \geq 1$. From the description that the $(j+1)$st m-dimensional simplex is formed by placing the jth m-dimensional simplex atop the $(j+1)$st $(m-1)$-dimensional simplex, find an equation expressing $F(j+1, m)$ in terms of $F(j, m)$ and $F(j+1, m-1)$. Since every zero-dimensional simplex is a point, set $F(j, 0) = 1$, $j \geq 1$. For what values of j and m does the recursion relation for $F(j+1, m)$ hold? Compare the numbers $F(j, m)$ to the arithmetical triangle of Blaise Pascal (1623–1662) [3, pp. 447–473], and compute a formula for $F(j, m)$ from your knowledge of this triangle. Pascal's genius was to organize the work of his predecessors on figurate numbers into one table.

(f) Let n be a positive integer, and consider the doubly nested loop

```
for i_1 = 1 to n
    for i_2 = 1 to i_1
        A
    next i_2
next i_1
```

Count the total number of times the statement A is executed by using a geometric argument. Suppose that A draws one dot upon every pass, next i_2 moves the cursor one space to the right, while next i_1 repositions the cursor to the far left, but on the next line. What values of j and m in $F(j, m)$ give the number of times the above double loop is iterated?

(g) Let n be a positive integer, and consider the triply nested loop

```
for i_1 = 1 to n
    for i_2 = 1 to i_1
        for i_3 = 1 to i_2
            A
        next i_3
    next i_2
next i_1
```

Count the number of times A is executed. Use a geometric argument similar to (f), except now next i_3 moves the cursor one space to the right, next i_2 repositions the cursor to the far left of the next line, and next i_1 moves the the cursor to the far left of the first line of the next plane. Find values of j and m so that $F(j, m)$ yields the number of times the above triple loop is iterated.

(h) Consider the k-fold nested loop, $k \geq 2$, $n \geq 1$,

```
for i_1 = 1 to n
    for i_2 = 1 to i_1
        for i_3 = 1 to i_2
            ⋮
                for i_k = 1 to i_{k-1}
                    A
                next i_k
            ⋮
        next i_3
    next i_2
next i_1
```

Let $L(n, k)$ be the number of times statement A is executed. Find a recursion relation for $L(n, k)$ in terms of previous values of L, and compare this to the recursion relation for $F(j, m)$. If A draws a dot, what figurate number is produced by the above program? Finally, find a formula for $L(n, k)$.

EXTRA CREDIT: Implement the above double and triple loops to actually draw the triangular and pyramidal numbers for various values of n. How might a three-dimensional object be best drawn on a two-dimensional sheet of paper?

References

[1] Katz, V. J., *A History of Mathematics: An Introduction*, Second Edition, Addison-Wesley, New York, 1998.

[2] Mahoney, M., *The mathematical career of Pierre de Fermat*, Second Edition, Princeton University Press, Princeton, New Jersey, 1994.

[3] Pascal, B., *Treatise on the Arithmetical Triangle*, Great Books of the Western World (Adler, M., editor), vol. 30, Encyclopaedia Britannica, Inc. 1991.